§3.6 Problems

These are some problems to practice the material above and do not represent homework unless explicitly mentioned otherwise. Give them a try! Some of them will be discussed by your TA during the upcoming discussion sessions from 4 to 5 PM on Tuesdays and Thursdays. I also included some solutions/sketches below.

Problem 3.6.1. Consider $mn+1$ closed intervals. Prove that either we can find $m+1$ pairwise disjoint intervals or we can find $n+1$ intervals with a nonempty intersection.

Problem 3.6.2. In a finite collection of intervals, among any $k+1$ we can find two with nonempty intersection. Prove that we can partition the collection into k subsets such that the intervals in each subset have pairwise nonempty intersections.

Problem 3.6.3. Given n sets, prove that we can choose at least $\sqrt{n}$ of them so that the union of no 2 of them is a third.

Problem 3.6.4. Suppose that $k \le \frac{2n}{3}$ and $A_1, ..., A_m$ are k-subsets of $[n]$ such that $A_i \cap A_j \cap A_k \neq \emptyset$ for all i, j, k. Prove that $m \le \binom{n-1}{k-1}$.

Problem 3.6.5. Let $A_1, ..., A_m \subset [n]$ be such that if $A_i \cap A_j = \emptyset$, then $A_i \cup A_j = [n]$. Prove that $m \le 2^{n-1} + \binom{n-1}{[\frac{n-2}{2}]}$.

Problem 3.6.6. Let $x_1, ..., x_n$ be positive real numbers, with $n > 1$. Show that there are less than $\frac{2^n}{\sqrt{n}}$ subsets A of $[n]$ such that $\sum_{i\in A} x_i = 1$.

Problem 3.6.7. Given 1001 rectangles with lengths and widths chosen from the set $\{1, 2, 3, \ldots, 1000\}$, prove that we can choose three of these rectangles, say A, B, C, such that A fits inside B and B fits inside C (rotations allowed).

Problem 3.6.8. The degree of a positive integer is the sum of the exponents of the primes in its prime factorization. Let $m \ge 2$ of degree n and let $d_1, ..., d_l$ be some positive divisors of m such that no d_i divides d_j with $i \neq j$. Then $l \le L$, where L is the number of divisors of m with degree $[n/2]$.

Problem 3.6.9. Let $k \le h \le n-k$ and let $A_1, ..., A_m$ be distinct subsets with k elements of $[n]$. Prove that we can find distinct subsets $B_1, ..., B_m$ with h elements of $[n]$ such that A_i and B_i are disjoint for all i.

Problem 3.6.10. Improve the second lemma in the proof of the EKR theorem as follows: suppose that $A_1, ..., A_m$ is an intersecting antichain in $[n]$, such that $|A_i| \le \frac{n}{2}$ for all i. Let σ be a cyclic permutation of $[n]$ and suppose σ contains A_i for some i. Prove that σ contains at most $|A_i|$ subsets among $A_1, ..., A_m$.

Problem 3.6.11. Let $A_1, ..., A_m$ be an intersecting antichain in $[n]$ such that $\max |A_i| \le \frac{n}{2}$.

a) Prove that for all cyclic permutations σ of $[n]$ we have

$$\sum_{A_i \subset \sigma} \frac{1}{|A_i|} \le 1.$$

b) Deduce the following theorem of Bollobas: if $A_1, ..., A_m$ is an intersecting antichain in $[n]$ such that $\max|A_i| \le \frac{n}{2}$, then

$$\sum_{i=1}^{m} \frac{1}{\binom{n-1}{|A_i|-1}} \le 1.$$

Problem 3.6.12. Let $k \le \frac{n}{2}$ and let $A_1, ..., A_m$ be an intersecting antichain in $[n]$ such that $|A_i| \le k$ for all i. Prove that $m \le \binom{n-1}{k-1}$. Moreover, if we have equality, then necessarily $|A_i| = k$ for all i.

Problem 3.6.13. Let an integer $n > 1$ be given. In the space with orthogonal coordinate system $Oxyz$ we denote by T the set of all points (x, y, z) with x, y, z are integers, satisfying the condition: $1 \le x, y, z \le n$. We paint all the points of T in such a way that: if the point $A(x_0, y_0, z_0)$ is painted then points $B(x_1, y_1, z_1)$ for which $x_1 \le x_0, y_1 \le y_0$ and $z_1 \le z_0$ is not be painted. Find the maximal number of points that we can paint in such a way the above mentioned condition is satisfied.

Problem 3.6.14. Given $k \le n$, find the largest m such that we can find m chains of $k+1$ distinct subsets of $[n]$ such that no member of any chain is a subset of a member of any other chain.

Problem 3.6.15. Let $k < n$ and let P be the set of subsets of $[n]$ that intersect $[k]$.

a) Prove the following theorem of Griggs: P is the union of chains, each of each is a symmetric chain in some symmetric chain decomposition of the subsets of S, or a symmetric chain minus its first element.

b) Prove that antichains in P have length at most $\binom{n}{[n/2]} - \binom{n-k}{[n/2]}$.